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Interpretation of Quantum Mechanics as a Theory of Extended Particles[1]

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Developing further the conclusions reached in an earlier paper, it is proposed to interpret quantum mechanics as a theory of extended particles. Certain restrictions are placed on the underlying model for extended particles. Wave–particle duality is interpreted in the context of the pulsations of the particle. The wave function is related to the (random) extension of the particle. It is shown that this wave function satisfies the Schrödinger equation. In this theory, the peculiarities of quantum probabilities are related to the assumption that the particle is shell-like. It is shown that a representation of dynamical variables by positive-operator-valued measures is possible. The empirical predictions of this theory are pointed out, along with some unsolved problems. It is concluded that it is, at least partially, possible to interpret quantum mechanics as a semiclassical description of the dynamics of extended particles. If this interpretation is correct, quantum mechanics would fail at very high energies, and, possibly, at very low energies.

1. INTRODUCTION

The problem of interpreting quantum mechanics is well known, and a review of the better-known interpretations can be found in Jammer (1974). More recently, a large class of (local) hidden variable theories have apparently been falsified (Clauser and Shimony, 1978), although the relevance of Bell's inequalities has been questioned—for instance, by Lochak (1977).

Here, we propose to adopt an altogether new approach. The logical basis of this approach is the assertion (Raju, 1980a) that an interpretation of the precise form of the indeterminacy relation necessarily leads to the

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following conclusions:

(i) The usual interpretation of the indeterminacy relations is fallacious within the axiomatic framework for quantum mechanics.

(ii) The particles (of nonzero rest mass) described by quantum mechanics cannot be localized, and, hence, must correspond to extended mass distributions.

It is therefore natural to think that the peculiar pattern of similarities and differences between classical and quantum mechanics arises because of the extended nature of real particles. In fact, by virtue of (ii), there is a definite theoretical necessity to interpret quantum mechanics as a theory of extended particles. In this paper, it is pointed out that many significant concepts of quantum mechanics have a natural counterpart in the context of a semiclassical description of the dynamics of extended particles. Looking at it in another way, the ideas presented below also have a direct bearing on the problem of describing the dynamics of extended particles in a manner that is Lorentz covariant and compatible with quantum mechanics.

2. THE MODEL FOR EXTENDED PARTICLES

The further development of this theory requires a model for extended particles. Apart from spherical symmetry in the rest frame, the main restrictions to be imposed on such a model would be the following:

(A) The particle interacts at its boundary with the external world.
(B) The particle pulsates uniformly in the rest frame.

Stated more mathematically, restriction (A) asserts that the mass of the particle is distributed (in the rest frame) over a spherically symmetric hypersurface, i.e., the particle is shell-like. From the point of view of relativity, such shell-like models seem to be necessary to overcome the classical phenomenological argument concerning the imbalance between gravitational and electromagnetic forces. A similar assumption has also been used in time-symmetric electrodynamics by Dirac (1938) and Raju (1980b), although the empirical consequences cannot be said to have been conclusively verified.

Such shell-like models have been constructed by Dirac (1962), for instance, by introducing new phenomenology to offset the inordinate imbalance between gravitational and electromagnetic forces. On the other hand, Raju (1981a) has proposed that such shell-like models can be obtained by suitably altering the usual junction conditions in relativity—if a surface layer of matter does not collapse under its self-action, there is no reason for a layer of charged matter to disintegrate. In fact, preliminary results indicate that charged surface layers can exist[2] and that oscillating solutions are possible. Although oscillating solutions also occur in Dirac's (1962) model, the advantage of this approach is that no new phenomenology is introduced.

Thus, we have a picture of an extended particle as an oscillating surface layer, obtained with or without additional phenomenology. With this picture we proceed with the interpretation of quantum mechanics.

Planck's Constant. The first step is to introduce an analog of the Planck constant into the theory, and connect it with the particular model, of extended particles, under consideration. This is done by defining a constant h_0 by

$$E_0 = h_0 \nu_0 \tag{1}$$

where E_0 is the energy, and ν_0 is the frequency of oscillation, measured in the rest frame. However, if (1) is to agree with the usual quantum mechanical relationship, some more restrictions are necessary on the model of extended particles:

(C) The oscillations of the particle are linear.
(D) The frequency of oscillation, ν_0, is proportional to the proper mass.
(E) The constant of proportionality is the Planck constant (with $c=1$).

These restrictions may seem to be too unreasonable for a realistic model of extended particles. However, with suitable generalizations of existing techniques (proposed in Raju, 1981a) preliminary results indicate that it may be possible to satisfy all the above restrictions, because the equations of motion for the surface layer turn out to be underdetermined. With Dirac's (1962) model, in the linear approximation, restrictions (A)–(D) are satisfied though (E) is not. The consequences of any possible nonlinearity are evaluated in Section 7.

Thus, in the context of extended particles, the wave–particle duality, implicit in (1), is interpreted as arising from the pulsations of the particle. Incidentally, we observe that the frequency of oscillation also leads to a

[2] More precisely, we are considering a hypersurface at the junction of the Reissner–Nordstrom and Minkowski metrics, described by the conditions that the components of the metric tensor be continuous across the hypersurface with essential discontinuities in some of their first derivatives. The results are rigorously derived by using nonstandard analysis to define products and compositions with the Dirac delta distribution (Raju, 1981b). A similar analysis when applied (Raju, 1981a) to the Schwarzschild–Minkowski junction, studied by earlier authors (Papapetrou and Hamoui, 1968, 1979; Evans, 1977), yields identical results.

Lorentz-covariant description of the energy of the center of mass of the particle.

3. THE WAVE FUNCTION

The next step is to introduce statistical considerations into the theory. This does not require any further restrictions because a particle in the real universe is never isolated. The external field, therefore, is at best statistically determined. For instance, the Brownian motion of the stars (Chandrashekhar, 1943) would induce fluctuations in the external gravitational field. Similarly, the random motion of nearby charged particles may be expected to produce small fluctuations in the external metric.

As a result of these fluctuations, the extension of the particle (i.e., the curvature of the spherical shell) and the phase of its oscillations, at any instant,[3] are random variables. We can combine the two to obtain a single, complex-valued random variable ϕ. $E4\pi|\phi|^2$ is, then, just the mean surface area of the particle ($|\phi|$ being the extension), and this must, presumably, be finite. Therefore, $\phi \in L^2 = L^2(\Omega, \mathbf{B}, P)$, $(\Omega, \mathbf{B}, P)$ being a standard Borel probability space. ϕ would be taken to correspond to the quantum mechanical wave function.

This assignment of a random variable to the state can be viewed classically in terms of incomplete information. But, because the entire cosmos is responsible for keeping this information incomplete, it is conceivable that it is impossible, even in principle, to have complete information about the state.

4. THE SCHRÖDINGER EQUATION

Suppose the particle is in equilibrium with its surroundings; then the random process $\phi(t)$ is stationary in the narrow sense, i.e., if $p(t_1, t_2, \ldots, t_k)$ represents the joint distribution of $\phi(t_1), \phi(t_2), \ldots, \phi(t_k)$ then

$$p(t_1+s, t_2+s, \ldots, t_k+s) = p(t_1, t_2, \ldots, t_k) \tag{2}$$

Equation (2) merely states that in statistical equilibrium a change of the time origin has no physical significance.

Because $\phi \in L^2$, stationarity in the narrow sense implies stationarity in the wide sense. That is, $E\phi(t)$ is independent of t, and the covariance $E\phi(t)\overline{\phi(s)}$ is a function only of the difference $t-s$. It follows (Rozanov,

[3]It is possible to speak of "the curvature of the shell at any instant," under the usual restriction that the normal to the shell be timelike.

1967) that the map $U(t)$, defined by

$$U(t)\phi(s)=\phi(t+s) \tag{3}$$

extends to a group of unitary operators on the subspace

$$H_{\infty}=\text{span}\{\phi(s), s\in\mathbb{R}\} \tag{4}$$

$U(t)$ extends trivially to L^2, and, by Stone's theorem, there exists a densely defined self-adjoint operator H_1 on L^2, such that

$$U(t)=\exp(-iH_1 t) \tag{5}$$

The operator H_1 satisfies the differential equation

$$\frac{\partial}{\partial t}U(t)=-iH_1U(t) \tag{6}$$

or, since $\phi(t)=U(t)\phi(0)$,

$$i\frac{\partial\phi}{\partial t}=H_1\phi \tag{7}$$

Let

$$H=h_0H_1 \tag{8}$$

where h_0 is defined by (1); then the spectrum of H just consists of the energy values of the particle, hence H is the Hamiltonian operator of quantum mechanics. H is trivially bounded below, and if h_0 is indeed the Planck constant then (7) is just the Schrödinger equation.

Thus, in the present theory, the Schrödinger equation appears as a consequence of (1) and some very general statistical laws.

5. QUANTUM PROBABILITIES AND THE OPERATOR REPRESENTATION

Wigner, in 1932, observed, and later proved (Wigner, 1972), that a joint distribution for position and momentum, consistent with linearity, does not exist in quantum mechanics. This observation has led to numerous attempts to formalize the notion of quantum probabilities, in the belief that they are significantly different from classical probabilities (in the sense of measure theory). On the other hand, one can very well adopt the point of view that Wigner's theorem asserts a failure of the operator representation, and that it is possible to formulate and work with quantum mechanics without introducing specifically "quantum" probabilities. Further, at the very heart of the problem lies the insistence that the position and momentum be random variables. In the extended particle case, this insistence may simply not be justified.

Before considering the extended particle case, we first consider some peculiarities of the probabilities appearing in quantum mechanics. For definiteness, we consider the probabilities regarding the ith position coordinate q of a single particle.

(i) The state ϕ itself generates a probability, since $\|\phi\|^2$ corresponds to the probability that q takes on some value.

(ii) Corresponding to each state ϕ we construct a random measure (i.e., a hilbert-space-valued measure) E_ϕ,

$$E_\phi(A)=E(A)\phi \tag{9}$$

E being the spectral measure induced by the self-adjoint operator corresponding to q.

(iii) Representing the state space as an L^2 space of random variables with zero means, we see that the probability $Q_\phi(A)$, that q takes on some value in the region $A\subseteq\mathbb{R}$

$$Q_\phi(A)=\frac{\|E(A)\phi\|^2}{\|\phi\|^2} \tag{10}$$

is essentially a ratio of the variances of two complex-valued random variables.

As Wiener (1958) has observed, a substantial part of the mystery of quantum mechanics lies in the fact that these peculiarities have never been satisfactorily explained.[4] This mystery can be resolved, at least partially, in the extended particle context. First, it is quite meaningless to speak of the "position" of the particle, since the particle simultaneously has several "positions." It might be a little more meaningful to speak of a portion of the particle lying in some region A. But, if the particle happens to be very small, say 10^{-30} cm across, a tremendous amount of energy would be required to resolve a portion of the particle. So, for all practical purposes, and for the range of energies for which quantum mechanics has been tested, it is quite meaningless to speak of a specific portion of the particle.

[4]Mathematically, a *single* probability measure, P, can always be represented as the variance measure of the random measure induced by a stationary stochastic process with covariance function $\hat{P}$, where $\hat{P}$ is the Fourier transform of P.

The next best thing one can do is to speak of the probability that the particle, on observation, would be found in some region A. Since observation involves an interaction with the external world, and since the particle interacts at its surface, this probability would be proportional to the total surface area of the particle in A. But, for a given region A, the surface area of the particle actually in A is a random variable dependent on the state, and its mean value would be taken to represent the above probability.

To summarize, one can speak meaningfully only of the probability of finding the particle in a certain region, and the probability of finding the ith position coordinate in region A is given by

$$P_\phi(A)=\frac{EX_\phi(A,\omega)}{EX_\phi(\mathbb{R},\omega)} \tag{11}$$

where $X_\phi(A,\omega)$ is the surface area of the sphere with center 0 (say) and radius $|\phi|$ that lies in the cylinder set over A in three dimensions. Thus, we see that $\|\phi\|^2$ is indeed proportional to the probability of finding the particle somewhere, and, to evaluate the probability of finding the ith position particle coordinate in A, we do have to construct a random measure. Moreover, as is customary in quantum mechanics, only the probability measure, and not a specific random variable distributed according to it, can be assigned to the dynamical variable.

Peculiarity (iii) requires a closer study. We first evaluate the probability $P_\phi(A)$. Since $EX_\phi(A)$ is additive, it is sufficient to evaluate this probability for regions A of the form $(-\infty, s]$. This is done in the Appendix and leads to the expression

$$\begin{aligned} EX_\phi(s,\omega)&=2\pi\sigma^2 e^{-s^2/\sigma^2}+\pi\sqrt{\pi}\,\sigma s\,\mathrm{Erfc}(-s/\sigma), && s<0\\ &=4\pi\sigma^2-2\pi\sigma^2 e^{-s^2/\sigma^2}+\pi\sqrt{\pi}\,\sigma s\,\mathrm{Erfc}(s/\sigma), && s>0 \end{aligned} \tag{12}$$

where $\mathrm{Erfc}(z)$ is the complementary error function. Using the asymptotic expansion

$$\mathrm{Erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-y^2}dy=\frac{e^{-z^2}}{\sqrt{\pi}\,z}\left\{1-\frac{1}{2z^2}+\cdots\right\} \tag{13}$$

(12) may be written in the form

$$\begin{aligned} EX_\phi(s,\omega)&=\pi\sigma^2 e^{-s^2/\sigma^2}\left\{1+\frac{\sigma^2}{2s^2}\cdots\right\}, && s<0\\ &=4\pi\sigma^2-\pi\sigma^2 e^{-s^2/\sigma^2}\left\{1+\frac{\sigma^2}{2s^2}-\cdots\right\}, && s>0 \end{aligned} \tag{14}$$

In arriving at (12), it has been assumed that ϕ has a complex Gaussian distribution with mean zero and variance σ^2. There is some virtue to the Gaussian distribution; however, the choice of a different distribution would have only a slight effect on the empirical consequences of the theory. Also, the parameter σ entering into (12) is not arbitrary, and is restricted by physical considerations. σ would be, approximately, at most half the order of magnitude of the mean extension of the particle. With the choice of a complex Gaussian distribution for ϕ, for example, σ would be of the same order of magnitude as the mean extension of the particle. In general, for a realistic model of, say, the electron, σ would be quite small.

We now claim that these probabilities do vary in an approximately quadratic manner with the state. More precisely, a map, $\phi \to \mu_\phi$, from the set of states to the set of finite positive measures, on the real line, would be said to vary quadratically if

1. $\mu_{\alpha\phi} = |\alpha|^2 \mu_\phi$
2. Parallelogram law:

$$\mu_{\phi+\xi} + \mu_{\phi-\xi} = 2\mu_\phi + 2\mu_\xi$$

3. $\mu_\phi(\mathbb{R}) = \|\phi\|^2$ (15)

with $\mu_\phi(A) = (1/4\pi) EX_\phi(A, \omega)$, we see that property 1, for instance, is satisfied if $|\alpha|$ is large compared to σ/s. Since σ/s is small, we can, for an approximate theory, assume that 1 is true for all values of α.

We now observe that, for each Borel set A, there exists a sesquilinear map

$$(\phi, \xi) \to \mu_{\phi,\xi}(A) \tag{16}$$

where

$$\mu_{\phi,\xi} = \tfrac{1}{4}\{\mu_{\phi+\xi} - \mu_{\phi-\xi} + i\mu_{\phi+i\xi} - i\mu_{\phi-i\xi}\} \tag{17}$$

is a complex Borel measure. Hence, there exist operators $E(A)$ such that

$$\mu_{\phi,\xi}(A) = \langle E(A)\phi, \xi\rangle \tag{18}$$

$E(A) \geqslant 0$, since

$$\langle E(A)\phi, \phi\rangle = \mu_{\phi,\phi}(A) = \mu_\phi(A) \geqslant 0 \tag{19}$$

If A_1 and A_2 are disjoint then

$$\begin{aligned}\langle E(A_1 \cup A_2)\phi, \phi\rangle &= \mu_\phi(A_1 \cup A_2)\\ &= \mu_\phi(A_1) + \mu_\phi(A_2)\\ &= \langle E(A_1)\phi, \phi\rangle + \langle E(A_2)\phi, \phi\rangle\\ &= \langle [E(A_1) + E(A_2)]\phi, \phi\rangle \end{aligned} \tag{20}$$

Since (20) holds for all ϕ

$$E(A_1 \cup A_2) = E(A_1) + E(A_2) \tag{21}$$

for disjoint A_1 and A_2. Further

$$E(\Phi) = 0 \tag{22}$$

trivially, and

$$\langle E(\mathbb{R})\phi, \phi\rangle = \mu_\phi(\mathbb{R}) = \|\phi\|^2 = \langle \phi, \phi\rangle = \langle I\phi, \phi\rangle \tag{23}$$

implies

$$E(\mathbb{R}) = I \tag{24}$$

where I is the identity operator.

It follows that $E(\cdot)$ is a positive-operator-valued measure. $E(\cdot)$ is a projection valued measure if

$$4. \quad \mu_{E(A_1)\phi, \xi}(A_2) = \mu_{\phi, \xi}(A_1 A_2) \tag{25}$$

An alternative formulation of condition 4, which is closer to the spirit of statistical mechanics, is obtained by noting that $E(\cdot)$ is a projection-valued measure iff $\sqrt{E}$ is additive, i.e., iff the random process

$$Y(s, \omega) = [E(-\infty, s)]^{1/2}\phi \tag{26}$$

is a martingale. These conditions are, however, not easy to interpret physically. On the other hand, since projections are weakly dense in the convex set of positive contractive operators, a reformulation of quantum mechanics in terms of positive-operator-valued measures (Davies, 1974) has certain advantages, including the existence of joint distributions.

6. FURTHER PROBLEMS

The correspondence proposed in the preceding sections indicates that an interpretation of quantum mechanics as a semiclassical description of the dynamics of extended particles is, at least partially, feasible. However, a final decision on the possibility of a complete interpretation of quantum mechanics, along these lines, must be deferred till the following issues are resolved:

(i) to determine whether the probabilities for momentum also vary quadratically, and to obtain, explicitly, the relationship between position, momentum and the Hamiltonian;
(ii) to determine whether such extended particles can have angular momentum with properties analogous to spin angular momentum;
(iii) to determine whether the behavior of such extended particles can be described in a Lorentz-covariant manner.

Although no definite solutions to these problems are available, at present, some possibilities are suggested below.

So far as (i) is concerned, we observe that the various (semiclassical) dynamical variables connected with the extended particle are, essentially, a little fuzzy around the corresponding classical values for the center of mass of the particle. Since one can obtain the Schrödinger equation from the Newtonian equations (for instance, Nelson, 1966), it is likely that the present methods would lead, approximately, to the usual equations of quantum mechanics, for a fairly large class of potentials. Slightly different methods would be required, however, since the present theory ascribes only a probability distribution, and not a specific random variable, to a dynamical variable.

Regarding (ii) we observe that there are the widespread misconceptions that intrinsic angular momentum is intrinsically quantum mechanical (for instance, Landau and Lifshitz, 1957, p. 186) and that a semiclassical explanation of spin must necessarily involve a rotating extended particle (for instance, McGregor, 1978). Intrinsic angular momentum can be defined for a classical "point" particle (Synge, 1966), and for an axisymmetric extended particle the net angular momentum need not be zero. Further, a semiclassical explanation for the Davisson–Germer experiment is impossible only if the structure of the dipole is assumed to be independent of the external field. Since the last assumption is quite false in the present theory, an explanation of spin may be difficult, but cannot be regarded as impossible, a priori.

(iii) does not appear to be an excessively difficult problem, since, by using a multicomponent wave function, an ellipsoid can be described in much the same way as a sphere.

7. EMPIRICAL TESTS

If this approach to the interpretation of quantum mechanics is correct, quantum theory would fail in certain situations. Quantitative predictions in concrete experimental situations may take some time to develop. Qualitatively, however, some situations, in which some of the axioms of quantum theory would fail, are immediately discernible.

(a) According to the present theory, the probabilities given by quantum mechanics are approximately correct for regions that are large compared to the mean extension of the particle. So, one can expect failures when $|s|$, in equation (5.4), is small. In practice, such situations would occur only when two particles interact at very high energies. It may not be feasible to test the other possibility, viz., that of σ being large.

(b) With extended particles, at high energies, departures from spherical symmetry are bound to occur. One way of testing this would be to look for quadrupole moments in the case of charged particles with spin.

(c) In case the oscillations of the particle are nonlinear, failures at low energies are also possible. In such a case, the pulsations of the particle may be considered as a superposition of oscillations at different frequencies. This would imply that the de Broglie relationship, $\lambda\nu = c^2/v$, is only approximately correct, and that other "wavelengths" can be associated with the particle, for the same value of the energy. These wavelengths would be observable as $\lambda \to \infty$, i.e., at low energies.

8. CONCLUSIONS

It is, at least partially, possible to interpret quantum mechanics as a semiclassical description of the dynamics of extended particles. If this interpretation is correct, then quantum mechanics would fail at very high, and, possibly, at very low energies.

APPENDIX

A. Evaluation of the Distribution Function. Let

$$X(s,\omega) = X\big((-\infty, s], \omega\big) \tag{A.1}$$

and

$$F_s(t)=P\{\omega, X(s,\omega)\leqslant t\} \tag{A.2}$$

Clearly

$$F_s(t)=0 \qquad \text{for } t<0, \tag{A.3}$$

and for $t\geqslant 0$,

$$\{\omega, X(s,\omega)\leqslant t\}=\{\omega, S\leqslant -|\phi|\}\cup\{\omega, -|\phi|\leqslant s\leqslant|\phi| \text{ and } 2\pi|\phi|(s+|\phi|)\leqslant t\}\cup\{\omega, |\phi|\leqslant s \quad \text{and } 4\pi|\phi|^2\leqslant t\} \tag{A.4}$$

since,

$$\begin{aligned} X(s,\omega)&=0 && \text{if } s\leqslant -|\phi|,\\ &=2\pi|\phi|(s+|\phi|) && \text{if } -|\phi|\leqslant s\leqslant|\phi|\\ &=4\pi|\phi|^2 && \text{if } |\phi|\leqslant s \end{aligned} \tag{A.5}$$

The probability of each of the sets in (A.4) is evaluated below.

(1)
$$\begin{aligned} P\{s\leqslant -|\phi|\} &= P\{|\phi|\leqslant -s\}\\ &=F_{|\phi|}(-s) \end{aligned} \tag{A.6}$$

where $F_{|\phi|}$ is the distribution function of $|\phi|$.

(2)
$$P\{|\phi|\leqslant s \quad \text{and} \quad 4\pi|\phi|^2\leqslant t\}=0 \qquad \text{if } s<0 \tag{A.7}$$

$$\left.\begin{aligned} &=F_{|\phi|}\left((t/4\pi)^{1/2}\right) && \text{if } t\leqslant 4\pi s^2\\ &=F_{|\phi|}(s) && \text{if } t\geqslant 4\pi s^2 \end{aligned}\right\} \quad s\geqslant 0 \tag{A.8}$$

(3) $P\{-|\phi|\leqslant s\leqslant|\phi| \quad \text{and} \quad 2\pi|\phi|^2+2\pi s|\phi|-t\leqslant 0\}$

The quadratic equation $2\pi x^2+2\pi sx-t=0$ always has two real roots since the discriminant $\Delta=4\pi^2s^2+8\pi t>0$, since $t>0$.

Let

$$s_{\pm}(t)=\frac{-2\pi s\pm(4\pi^2 s^2+8\pi t)^{1/2}}{4\pi}$$

$$=-s/2\pm\tfrac{1}{2}(s^2+2t/\pi)^{1/2} \qquad \text{(A.9)}$$

then $2\pi|\phi|^2+2\pi s|\phi|-t\leqslant 0$ iff

$$s_-\leqslant|\phi|\leqslant s_+ \qquad \text{(A.10)}$$

If $s\geqslant 0$ then

$$s_-=\frac{s}{2}-\frac{1}{2}\left(s^2+\frac{2t}{\pi}\right)^{1/2}\leqslant-\frac{s}{2}\leqslant s \qquad \text{(A.11)}$$

and $s_+\geqslant s$ if

$$-\frac{s}{2}+\frac{1}{2}\left(s^2+\frac{2t}{\pi}\right)^{1/2}\geqslant s$$

i.e., iff

$$t\geqslant 4\pi s^2 \qquad \text{(A.12)}$$

Thus the above probability is zero if $t\leqslant 4\pi s^2$ and is otherwise

$$F_{|\phi|}(s_+)-F_{|\phi|}(s) \qquad \text{(A.13)}$$

If $s\leqslant 0$ then

$$s_-=-\frac{s}{2}-\frac{1}{2}\left(s^2+\frac{2t}{\pi}\right)^{1/2}\leqslant-\frac{s}{2}\leqslant-s \qquad \text{(A.14)}$$

and

$$s_+\geqslant -s, \qquad \text{since } t>0. \qquad \text{(A.15)}$$

It follows that the above probability is $F_{|\phi|}(s_+)-F_{|\phi|}(-s)$ for all $t\geqslant 0$. Thus

$$\left.\begin{aligned} F_s(t)&=F_{|\phi|}\left((t/4\pi)^{1/2}\right) && \text{if } t\leqslant 4\pi s^2\\ &=F_{|\phi|}(s_+(t)) && \text{if } t>4\pi s^2\end{aligned}\right\}\quad s\geqslant 0 \qquad \text{(A.16)}$$

and $$F_s(t)=F_{|\phi|}(s_+(t)) \qquad \text{if } s\leqslant 0 \tag{A.17}$$

$$EX_\phi(s,\omega)=\int_0^\infty t\,dF_s(t) \tag{A.18}$$

has been evaluated below assuming that ϕ has a complex Gaussian distribution with mean zero and variance σ^2.

B. Evaluation of the Integral

$$I=\int_0^\infty t\,dF_s(t) \tag{B.1}$$

Case I. If $s<0$, then

$$F_s(t)=F_{|\xi|}(s_+(t)) \tag{B.2}$$

where

$$dF_{|\xi|}(x)=(1/\sigma^2)xe^{-x^2/2\sigma^2}\,dx, \qquad x\geqslant 0$$
$$=0, \qquad x\leqslant 0 \tag{B.3}$$

and $s_+(t)$ is defined by (A.9) and for simplicity we take $\|\xi\|_2^2=2\sigma^2$.

Integrating by parts, making the transformation

$$t\to v=+(s^2+2t/\pi)^{1/2} \tag{B.4}$$

and observing that when $t=0$, $v=-s$ since $s<0$

$$I=\int_{-s}^\infty \pi v e^{-(v-s)^2/8\sigma^2}\,dv \tag{B.5}$$

with

$$v\to z=v-s$$

$$I=4\pi\sigma^2 e^{-s^2/2\sigma^2}+\pi\sqrt{\pi}\,\sqrt{2}\,\sigma s \qquad \mathrm{Erfc}\left(-s\big/-\sqrt{2}\,\sigma\right) \tag{B. 6}$$

Case II. If $s\geqslant 0$, then

$$F(t)=F_{|\xi|}\left((t/4\pi)^{1/2}\right) \qquad \text{if } t\leqslant 4\pi s^2$$
$$=F_{|\xi|}(s_+(t)) \qquad \text{if } t\geqslant 4\pi s^2 \tag{B.7}$$

with $F_{|\xi|}$ and $s_+(t)$ as before and

$$I=I_1+I_2 \tag{B. 8}$$

$$I_1=\int_0^{4\pi s^2} t dF_{|\xi|}\left((t/4\pi)^{1/2}\right)$$

$$=-4\pi s^2 e^{-s^2/2\sigma^2}+8\pi^{\sigma^2}\left(1-e^{-s^2/2\sigma^2}\right) \tag{B. 9}$$

and

$$I_2=\int_{4\pi s^2}^{\infty} t dF_{|\xi|}(s_+(t)) \tag{B. 10}$$

reduces as in case I above to give

$$I_2=4\pi s^2 e^{-s^2/2\sigma^2}+4\pi^{\sigma^2} e^{-s^2/2\sigma^2}$$
$$+\pi\sqrt{\pi}\sqrt{2}\sigma \text{ Erfc}\left(s/\sqrt{2}\sigma\right) \tag{B. 11}$$

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